

NOTES ON ERGODIC THEOREMS IN NON-COMMUTATIVE SYMMETRIC SPACES

GENADY YA. GRABARNIK

ABSTRACT. In this paper we establish individual ergodic theorem for positive kernels (or so called Danford Shwartz (DS+) operators acting on non commutative symmetric spaces.

1. INTRODUCTION

The goal of the paper is to see that one of the results by Veksler [20] or Muratov, Pashkova and Rubshtein [16] remains valid for the non commutative case.¹ Let M, τ be a semifinite von Neumann algebra with semifinite normal faithful trace τ .

In addition we assume that M, τ satisfy homogeneity property, it may be presented in the form of the resonant property on trace, see for example [1].

The space of all measurable operators affiliated with M, τ in the Sigal [19] sense is denoted by L_0 , see for details [19, 5, 6].

Notions of $L_1(M, \tau)$ and $L_\infty(M, \tau)$ was naturally introduced in the same paper.

Since we fix algebra M and trace τ , we omit them from the notations from now on.

Let $F \in M$ be a set of finite linear combinations of orthogonal projections with finite trace.

Space $R_0 = (L_1 + L_\infty)_0$ is the closure of F in the norm $\|x\| = \inf\{\|x_1\|_1 + \|x_2\|_\infty, |x = x_1 + x_2, x_1 \in L_1, x_2 \in L_\infty\}$.

Remark 1. *The space R_0 is not necessary separable, it is sufficient to consider $M = B(H)$ of all bounded operators in the Hilbert space H with not separable H with natural trace τ .*

Definition 1. *Non-commutative re-arrangement invariant (or symmetric) space L for the fully symmetric case were introduced by Yeadon in [24]. For the definition of the symmetric spaces we refer to the recent book of Lord, Sukochev, Zanin on singular traces, with original proofs due to Kalton and Sukochev.*

Definition 2. *Re-arrangement invariant space L over (M, τ) is called minimal if F is dense in L by norm of L .*

Date: March 1, 2016.

1991 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Ergodic theory, Operator algebras, re-arrangement invariant spaces.

¹ It become known to author that Litvinov and Chilin also saw this result at the same time and wrote it at the same time. I suggested to them to combine results and names on paper. Waiting for the answer.

2. EMBEDDING THEOREM

The following refinement of the embedding theorem [12] take place.

Theorem 1. *Let L be a re-arrangement invariant space over (M, τ) . Then*

(1) *If L is minimal, then*

$$L_1 \cap L_\infty \subseteq L \subseteq R_0$$

(2) *If L is not minimal L , then*

$$L_\infty \subseteq L \subseteq L_1 + L_\infty$$

Proof. Proof is given in the forthcoming paper of author and some co-authors. \square

3. INDIVIDUAL ERGODIC THEOREM FOR MINIMAL SYMMETRIC SPACES

3.1. The Largest Minimal Symmetric Space. The largest minimal symmetric space is $R_0 = (L_1 + L_\infty)_0$ is a set of all measurable integrable with trace operators plus bounded with not increasing re-arrangement functions decreasing to 0.

The space R_0 is minimal symmetric space. Any minimal symmetric space is a subset of R_0 .

3.2. Positive double contraction on R_0 . Any positive kernel ($T \in DS^+$) leaves R_0 invariant.

3.3. Mean Ergodic Theorem. Von Neumann Mean ergodic Theorem on L_2 follows from the general von Neumann ergodic theorem for the contractions on the general Hilbert spaces.

Mean convergence on $L_1 \cap L_\infty$ follows from the the fact that both L_1 and L_∞ are invariant under positive kernels. The space $L_1 \cap L_\infty$ itself is invariant under action of positive kernel, and positive kernel is a contraction of the space $L_1 \cap L_\infty$. The von Neumann Ergodic theorem and the closedness of $L_1 \cap L_\infty$ in its norm implies mean ergodic theorem on $L_1 \cap L_\infty$.

3.4. Mean convergence on R_0 .

Proposition 1. *The Cesaro averages $S_n(T)x$ converge to some \tilde{x} in norm of R_0 .*

Proof. Follows from von Neumann ergodic theorem and the fact that L_2 is dense R_0 . Details. We show that $s_l(T)x, l = 1, 2, \dots$ is fundamental sequence in R_0 . Indeed, each $x \in R_0$ may be presented as $x = x_{1,n} + x_{2,n}$ with $x_{1,n} \in L_1$ and $x_{2,n} \in L_\infty$ with $\|x_{2,n}\|_\infty < 2^{-n}$. Then we can apply von Neumann or Yeadon's Mean Ergodic theorem 4.2 [24] for L_1 to $x_{1,n}$ and find $l(n)$ such that for $l, m \geq l(n)$ holds

$$\|s_l(T)x_{1,n} - s_m(T)x_{1,n}\|_{L_1} < 2^{-n}.$$

Then for $l, m \geq l(n)$

$$\begin{aligned} \|s_l(T)x - s_m(T)x\|_{R_0} &\leq \|s_l(T)x_{1,n} - s_m(T)x_{1,n}\|_{L_1} + \\ &+ \|s_l(T)x_{2,n} - s_m(T)x_{2,n}\|_{L_\infty} \leq 4 * 2^{-n}, \end{aligned}$$

and, hence, the sequence $s_l(T)x$ is fundamental. Completeness of R_0 implies existence of $\tilde{x} \lim_{l \rightarrow \infty} s_l(T)x$. \square

Remark 2. Mean ergodic theorem for fully symmetric spaces is due to Yeadon [24], Theorem 4.2. Note that we do not require the space L to be fully symmetric here. Condition ii) of the theorem 4.2 [24] means that the space L is minimal. The space R_0 does not satisfy condition iii) in theorem 4.2 [24].

3.5. Individual Ergodic Theorem in L_1 . Individual Ergodic theorem for L_1 was established by Yeadon [22], among other authors.

3.6. Individual ergodic theorem for R_0 . The goal of the section is to show double side almost everywhere convergence for the operators from the R_0 .

Definition 3. The sequence x_n from L_0 is called converging double side almost everywhere to $x_0 \in L_0$ if for every $\epsilon > 0$ there exist orthogonal projection $E \in M$ such that $\tau(1 - E) < \epsilon$, $E(x_n - x_0)E \in M$ and $E(x_n - x_0)E \rightarrow 0$.

Theorem 2. Let M, τ, R_0 are as above and T is positive kernel on M . For $x \in R_0$, Cesaro averages $S_n(T)x$ converge d.s.a.e. in R_0 .

Proof. The proof follows the line of the proof of Proposition 1 and uses Yeadon's individual ergodic theorem for L_1 [22], see also Chilin, Litvinov [?], [?]

We show that $s_l(T)x, l = 1, 2, \dots$ is fundamental d.s.a.e. sequence in R_0 . Indeed, each $x \in R_0$ may be presented as $x = x_{1,n} + x_{2,n}$ with $x_{1,n} \in L_1$ and $x_{2,n} \in L_\infty$ with $\|x_{2,n}\|_\infty < 2^{-4*n}$. In turn, $x_{1,n} = x_{1,1,n} + x_{1,2,n}$, with $x_{1,1,n} \in L_\infty$, $x_{1,2,n} \in L_1$ and $\|x_{1,2,n}\|_{L_1} < 2^{-8*n}$, $n = 1, 2, \dots$

Then we can apply Yeadon's Individual Ergodic theorem 1, [22] for L_1 to $x_{1,1,n}$ and find $l(n)$ and projector $E(n) \in M$ such that $\tau(1 - E(n)) < 2^{-4*n}$ and for $l, m \geq l(n)$ holds

$$\|E(n)(s_l(T)x_{1,1,n} - s_m(T)x_{1,1,n})E(n)\|_{L_\infty} \rightarrow 0.$$

We can represent $x_{1,2,n} = \sum_{k=1}^\infty x_{2,n,k}$, with $x_{2,n,k} \in L_\infty$ and $\|x_{2,n,k}\|_{L_1} < 2^{-8*(n+k)}$.

Then, we can find $E(1, n) = \bigwedge_k E(1, n, k)$, with $\tau(1 - E(1, n)) \leq 2^{-4*n}$ and $E(1, n)s_l(x_{1,2,n})E(1, n) \in L_\infty$ and $\|E(1, n)s_l(x_{2,n})E(1, n)\|_{L_\infty} < 2^{-4*n}$, where projections $E(1, n, k)$ are obtain by Theorem 1 from [22] applied to $x_{2,n,k}$ and $\epsilon = 2^{-4(n+k)}$.

By choosing $E(2, n) = \bigwedge_{k=1}^\infty E(1, n+k) \wedge \bigwedge_{k=1}^\infty E(n+k)$, we have $\tau(1 - E(2, n)) < 2^{-4*n}$.

Moreover, for $l, m \geq l(n)$

$$\begin{aligned} & \|E(2, n)(s_l(T)x - s_m(T)x)E(2, n)\|_{L_\infty} \\ & \leq \|E(2, n)(s_l(T)x_{1,n} - s_m(T)x_{1,n})E(2, n)\|_{L_\infty} + \|E(2, n)(s_l(T)x_{2,n} - s_m(T)x_{2,n})E(2, n)\|_{L_\infty} \\ & \leq \|E(2, n)(s_l(T)x_{1,1,n} - s_m(T)x_{1,1,n})E(2, n)\|_{L_\infty} + \|E(2, n)(s_l(T)x_{1,2,n}E(2, n)\|_{L_\infty} \\ & \quad + \|E(2, n)s_m(T)x_{1,2,n}E(2, n)\|_{L_\infty} + 2 * 2^{-4*n} \leq 8 * 2^{-n}, \end{aligned}$$

and, hence, the sequence $s_l(T)x$ is fundamental d.s.a.e. The sequence $s_l(T)x$ also converges in norm in R_0 to \tilde{x} , hence it converges in measure. This implies convergence of $s_l(T)x$ d.s.a.e. to \tilde{x} . \square

Remark 3. In the case when space L is not minimal, it contains M . Then it is possible to show [16], that even in the commutative case, there exists ergodic automorphism of the space with measure such that ergodic averages do not converge almost everywhere.

Corollary 1. *Let L be a minimal non-commutative symmetric space. Let T be a positive kernel such that T leaves L invariant and T acts as contraction on L . Then Cesaro averages $s_n(T)x$ converge in norm and d.s.a.e. for any $x \in L$.*

Proof. Since the space L is minimal, the set of $L_1 \cap L_\infty$ is dense in L . Since $L_1 \cap L_\infty \subseteq L$, hence

$$\|x\|_{L_1 \cap L_\infty} \geq C * \|x\|_L$$

for any $x \in L_1 \cap L_\infty$, which in turn implies convergence of Cesaro averages of $s_n(T)x$ in norm of L for $x \in L_1 \cap L_\infty$. Fix real $\epsilon > 0$. Find $x_k \in L_1 \cap L_\infty$ with $\|x - x_k\|_L < \epsilon/2$. Then Cesaro averages are $s_n(T)x$ are within ϵ of the \tilde{x}_k , where $\tilde{x}_k = \lim_{n \rightarrow \infty} s_n(T)x_k$. This implies norm convergence of $s_n(T)x$.

The d.s.a.e. convergence follows from the embedding theorem 1, since a minimal re-arrangement invariant non-commutative function space L is a subspace of R_0 . \square

Corollary 2. *Let L be a minimal fully symmetric space. Let T be a positive kernel on (M, τ) . Then T leaves L invariant and act on L as contraction. Moreover, the Cesaro averages $S_n(T)x$ converge d.a.e. for any $x \in L$.*

Corollary 3. *(see Chilin Litvinov [3]). Let L_Ψ be a non-commutative Orlicz space with function Ψ satisfying conditions δ_2 and Δ_2 . Let T be a positive kernel. Then the Cesaro averages $S_n(T)x$ converge d.s.a.e. for any $x \in L$.*

Proof. The Orlicz space L_Ψ with function Ψ satisfying conditions δ_2 and Δ_2 is minimal [1, 12, 15]. Since the space L_Ψ is fully symmetric, it is interpolation space [1, 12] and hence T leaves L_Ψ invariant and acts on L_Ψ as a contraction. Then we are in the assumptions of the Corollary 2, and hence $S_n(T)x$ converges d.a.e. . \square

REFERENCES

- [1] C. Bennett, R. Sharpley, **Interpolation of Operators**, Academic Press Inc. (London) LTD, 1988.
- [2] V. Chilin, S. Litvinov, On pointwise ergodic theorems for infinite measure, arhiv, 2015
- [3] V. Chilin, S. Litvinov, Individual ergodic theorems in noncommutative Orlicz spaces, arhiv, 2016
- [4] V. I. Chilin, F. A. Sukochev, Weak convergence in non-commutative symmetric spaces, *J. Operator Theory*, 31 (1994), 35-65.
- [5] P. G. Dodds, T. K. Dodds, and B. Pagter, Fully symmetric operator spaces, *Integr. Equat. Oper. Theory*, 15 (1992), 942-972.
- [6] P. G. Dodds, T. K. Dodds, and B. Pagter, Noncommutative Kothe duality, *Trans. Amer. Math. Soc.*, 339(2) (1993), 717-750.
- [7] P. G. Dodds, T. K. Dodds, F. A. Sukochev, and O. Ye. Tikhonov, A Non-commutative Yoshida-Hewitt theorem and convex sets of measurable operators closed locally in measure, *Positivity*, 9 (2005), 457-484.
- [8] P. G. Dodds, B. Pagter and F. A. Sukochev, Sets of uniformly absolutely continuous norm in symmetric spaces of measurable operators, *Trans. Amer. Math. Soc.*, (2015).
- [9] N. Dunford and J. T. Schwartz, **Linear Operators, Part I: General Theory**, John Willey and Sons, 1988.
- [10] T. Fack, H. Kosaki, Generalized s -numbers of τ -mesaurable operators, *Pacific J. Math.*, 123(1986), 269-300.
- [11] R. V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. of Math.* (2), 56(1952), 494-503.
- [12] S. G. Krein, Ju. I. Petunin, and E. M. Semenov, **Interpolation of Linear Operators**, *Translations of Mathematical Monographs*, Amer. Math. Soc., **54**, 1982.
- [13] J. Lindenstrauss, L. Tsafriri, **Classical Banach spaces I-II**, Springer-Verlag, Berlin Heidelberg New York. 1977.

- [14] E. Nelson, Notes on non-commutative integration, *J. Funct. Anal.*, 15 (1974), 103-116.
- [15] B.Z. Rubshtein, G. Ya. Grabarnik, M. A. Muratov, Pashkova, SSMF, in press, 2017
- [16] M. A. Muratov, J. Pashkova, B-Z. Rubshtein Order Convergence Ergodic Theorems in Rearrangement Invariant Spaces, Operator Methods in Mathematical Physics: Conference on Operator Theory, Analysis and Mathematical Physics, 2013
- [17] B.Z. Rubshtein, M. A. Muratov, Pashkova, Embedding theorem, oral communication, 2017
- [18] S. Sakai, *C^* -algebras and W^* -algebras*, Springer-Verlag, Berlin Heidelberg New York, 1971.
- [19] I. E. Segal, A non-commutative extension of abstract integration, *Ann. of Math.*, 57 (1953), 401-457.
- [20] A. Veksler, An ergodic theorem in symmetric spaces, *Subirsk. Mat. Zh.*, 24 (1985), 189-191 (in Russian).
- [21] H. Yanhou, T. N. Bekjan, The dual on noncommutative Lorentz spaces, *Acta Math. Sci.*, 31 B(5)(2011), 2067-2080.
- [22] F. J. Yeadon, Non-commutative L^p -spaces, *Math. Proc. Camb. Phil. Soc.*, 77(1975), 91-102.
- [23] F. J. Yeadon, Ergodic theorems for semifinite von Neumann algebras-I, *J. London Math. Soc.*, 16 (2)(1977), 326-332.
- [24] F. J. Yeadon, Ergodic theorems for semifinite von Neumann algebras: II, *Math. Proc. Camb. Phil. Soc.*, 88 (1980), 135-147.

(GYaG) ST JOHNS UNIVERSITY, QUEENS, NY, USA
E-mail address, GYaG: grabarng@stjohns.edu